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Multi-Dimensional Quantified Modal Logic

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The most familiar modal logics are one-dimensional in the sense that they are characterized by models in which sentences are evaluated at individual worlds. But while one-dimensional modal logics may be most familiar, they also have certain limitations. And those limitations have convinced many of us that important modal notions—like metaphysical necessity and physical necessity—cannot be fully characterized in a one-dimensional framework. What is needed, then, is a multi-dimensional framework, one in which sentences are evaluated at *sequences* of worlds rather than just worlds.

The plan for this paper is as follows. In section one, we will survey some of the motivations for multi-dimensional modal logic. We are then going to specify a hierarchy of propositional modal languages (PMLs) in section two and a corresponding hierarchy of quantified modal languages (QMLs) in section three. Along the way, we are going to generalize the familiar notion of a Kripke model to arbitrarily many dimensions. This is meant to correct what I see as a certain problem in the literature. There, the most commonly used multi-dimensional models validate all the axioms of S5. Even when more general models are used, they still validate axioms that are less than ideal in certain contexts. For example:

$$\phi \supset \Diamond\phi$$

But now suppose that we are interested in building a multi-dimensional *deontological* logic, and so want to think of the diamond operator as expressing permission. In that case, this tells us that whatever is actual is also permissible, which is clearly absurd. And so if multi-dimensional modal logic is to have its widest range of application, we are going to need a more general framework.

After describing a hierarchy of languages and models, we are going to turn our attention to proof theory. For simplicity and concreteness, we are going to focus on the two-dimensional case, but the basic approach extends to higher dimensions as well. Section four describes several systems, the most important of which are $\mathbf{K_2}$ and $\mathbf{Q_2}$. The first is the logic of all two-dimensional Kripke models. The second is the logic of all *quantified* two-dimensional Kripke models. That this is so will be shown in sections five and six, where we provide completeness results for both systems. To my knowledge, there are no published completeness results for quantified multi-dimensional modal logic. And so this second result is meant to fill what I see as an important gap in the literature.

1. Motivation

The language of quantified predicate logic (QPL) was built to regiment the practice of mathematics. But the practice of mathematics, being what it is, is a purely categorical enterprise. It is about how certain mathematical structures are, not how they *could* have been, *should* have been, or *were*. Ordinarily, though, we think that there are facts involving such notions. We think that there *could* have been more planets, that there *should* have been less suffering, and that there *were* once dinosaurs who roamed the earth. This then raises a natural question: Given that there are such facts, what sort of formal language should we use to express them?

There are two basic approaches. The **regimented** approach says that despite initial appearances, the language of quantified predicate logic is the right one after all.[1] All we need is quantification over the right sort of indexes. In the modal case—which will be our main focus here—those indexes are possible worlds.[2] To fix on an example, proponents of the regimented approach might use a two-sorted quantified predicate language $\mathcal{E}$ with one stock of variables w, v, u... for possible worlds and another x, y, z... for the individuals who inhabit those worlds. Names and argument places are then similarly sorted. The logical predicates of the language are:

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1. My use of the regimented/autonomous terminology is borrowed from Burgess (2009). The distinction is similar to—but also distinct from—the modalist/possiblist distinction from chapter one. In particular, the autonomous/regimented distinction is purely linguistic. Neither view makes any particular claims about the nature of reality. The modalist/possiblist dispute, on the other hand, is meant to be metaphysical. The modalist thinks that modality is ultimately about how things could have been. The possiblist thinks that modality is ultimately about what there is. Generally speaking, modalists are aligned with the autonomous approach and possiblists with the regimented approach. But you could also imagine various non-standard ways in which the views might be combined.

2. For simplicity, we are going to generally think of the regimented approach as an *indexing* approach, meaning that it quantifies over possible worlds and adds argument places to predicates. There are also other strategies, though, that you might pursue. You might, for example, follow Lewis 1968 and use quantification over counterparts instead of quantification over worlds.

Aw	w is the actual world
Ixw	x is in w
$x = y$	x is identical to y

The actuality predicate takes terms for worlds. The in predicate takes terms for individuals in the first position and terms for worlds in the second. The identity predicates takes terms of either sort in both positions. Modal facts are then expressed by indexing non-logical predicates to worlds. So for example, we could say that there is a possible world at which Socrates is a farmer by writing $\exists w Fsw$.

The second approach is the **autonomous** approach. The autonomous approach says that intensional notions are best expressed by extending the language of QPL with certain sentential operators. In the modal case, the standard autonomous language is the language of one-dimensional quantified modal logic, which we are going to call $\mathcal{Q}_1$. That language is built by adding a necessity operator $\Box$ to the language of QPL and then defining a possibility operator $\Diamond$ in terms of it. This lets us express the idea that Socrates could have been a farmer by writing $\Diamond Fs$.

In many cases, we can **directly translate** between the regimented language $\mathcal{E}$ and the autonomous language $\mathcal{Q}_1$, For example:

$$\exists w \exists x (Ixw \wedge Fsw) \mapsto \Diamond \exists x Fx$$

If I say that there is a possible world at which someone is a farmer, you can translate that by saying that it could have been that someone was a farmer. This is a direct translation in the sense that it directly trades quantification over worlds for operators. You can even see how the translation might be the result of a certain purely *mechanical* (and in fact recursive) procedure. That is, to get from the first sentence to the second, we just (a) exchange the existential quantification over worlds for a diamond, (b) exchange the restricted quantification over individuals for unrestricted quantification, and (c) delete the the extra argument place from the *is a farmer* predicate. This sort of procedure could be made fully precise. But you get the basic idea.[3]

The problem for the autonomous approach is that there are natural modal claims that can be expressed in $\mathcal{E}$ that have no direct translation into Q_1. One kind of familiar example is from Hazen (1976). Ordinarily, we think that:

> There could have been something that does not actually exist.

This fact can be easily expressed in the regimented language. We can just say that:

$$\exists v(Av \wedge \exists w \exists x(Ixw \wedge \forall y(Ixv \supset x \neq y)))$$

The actual world is v and there is a possible world w such that some x in w is not identical to anything in v. The problem is that there is no way to directly translate this sentence into Q_1, and so no way to directly express the idea that there could have been something that does not actually exist.

A second kind of case involves relations across worlds. Suppose, for example, that Aristotle is taller than Socrates. This fact is naturally expressed in quantified predicate logic by writing Tas. But now suppose that we want to say that:

3. These sorts of direct translations are sometimes called *standard translations* and are the subject matter of correspondence theory. For more on this, see the second chapter of Blackburn, Rijke, and Venema (2010). For some examples of multi-dimensional standard translations, see Marx and Venema (1997).

Aristotle could have been taller than Socrates actually is.

This is naturally expressed in the regimented language by using a double-indexed *taller than* predicate:

$$\exists v(Av \wedge \exists w(Tawsv))$$

The actual world is v and that there is a possible world w such that Aristotle at w is taller than Socrates at v. The problem for the autonomous approach is that again, there is no way to directly translate this perfectly sensible claim into $\mathcal{Q}_1$.

As it turns out, these two sorts of problems are picking up on the same structural feature. Namely: $\mathcal{Q}_1$ can only translate sentences of $\mathcal{E}$ that have a single world variable.[4] But both of the above claims require at least *two* world variables. And so there is no way of directly translating them into $\mathcal{Q}_1$. Or putting it somewhat more intuitively: $\mathcal{Q}_1$ is a language for describing the intrinsic features of worlds. But ordinarily, we think that there are certain kinds of *comparisons* and *relations* between worlds. These are easy to describe in $\mathcal{E}$. But those descriptions then have no direct translation into $\mathcal{Q}_1$.

Once you see the basic structural problem, it is easy to find other instances. There is,

4. While this is *mostly* true, there is also an exception involving the accessibility relation. Suppose, for example, that we add an accessibility predicate to $\mathcal{E}$. If our modal operators are also appropriately restricted by accessibility, we can then translate $\exists v(Av \wedge \exists w(Rwv \wedge Fsw))$ as $\Diamond Fs$. But this sentence requires two world variables so, strictly speaking, it is not true that $\mathcal{Q}_1$ cannot translate such sentences. What is true is that it can translate sentences that require two world variables only if the second world variable is only needed to express facts involving the accessibility relation. This sort of limited exception, though, will not help with the problematic cases.

for example, a certain family of views that go under the heading of **modal relativism**. The modal relativist about properties, for example, says that whether or not something has a property at a particular world is itself a relative matter. So for example, suppose that there is a particular drinking glass g at a possible world w that is filled with H_2O. Does that glass also have the property of being filled with water? The modal relativist about properties says that the answer depends on which world is actual. Relative to our world v being actual, g is filled with water at w, since our world is one in which H_2O plays the water role. But relative to a different world u in which some other substance XYZ plays the water role, g is not filled with water at w. This basic idea is easy to express using $\mathcal{E}$. We can just use a double-indexed *is filled with water* predicate:

$$\exists v(Av \wedge \exists w(Wgwv \wedge \exists u Wgwu))$$

But because this claim require two world variables, there is no way to express it in Q_1. Moreover, besides modal relativism about properties, there are other sorts of views you might want to hold: You might want to be a modal relativist about *ontology*. You might think that certain things exist at a world w from the perspective of one world v, but not from the perspective of some other world u.[5] Or you might be a modal relativist about *modality*. You might think that in certain cases, a world w is accessible from v from the perspective of x, but not accessible from v from the perspective of some other world y.[6]

The idea behind multi-dimensional modal logic is to build autonomous languages

5. See my Berntson (2019) for a defense of this sort of view.

6. See Murray and Wilson (2012) and Helle, Murray, and Wilson (2018) for more on modal relativism.

with fewer restrictions. In the next section, we are going to build a whole hierarchy of modal languages Q_n, with each having the ability to directly translate claims that require n world variables. Most of the problematic cases require only two world variables, so can be expressed in Q_2. For example:

$\Diamond\exists x{\downarrow}\forall y(x \neq y)$

$\Diamond Tas$

$\Diamond\otimes Tas$

$\Diamond\uparrow Tas$

$\Diamond(Wg \wedge \otimes\Diamond\otimes\neg Wg)$

The first sentence says that there could have been something that does not actually exist. The second says that Aristotle could have been taller than Socrates actually is. The third says that Aristotle could have been *shorter* that Socrates actually is. The fourth says that Aristotle could have been taller than Socrates. And the fifth says that there is a possible world at which a glass is filled with water, relative to the actual world, even though it is not filled with water relative to some other possible world.

Now of course, there are also other strategies for dealing with the expressive limitations of Q_1. You might, for example, try to use certain **indirect translations** in place of direct translations. In the first case, you might try expressing the idea that there could have been something that does not actually exist by quantifying over sets:

$\exists x(\forall y(y \in x) \wedge \Diamond\exists z(z \notin x))$

There is a set that has everything as a member and it could have been that something existed without being a member of that set. In the second case, you might try to express the idea that Socrates could have been taller than he actually is by quantifying over heights:

$$\exists x(Hax \wedge \Diamond \exists y(Hsy \wedge y > x))$$

There exists a height that Aristotle has and it could have been that there was a height Socrates had that was greater than the one Aristotle has. Given the right background assumptions, you might think, these represent perfectly good translations of the corresponding claims.

The problem with indirect translations is that they almost always come with additional commitments. In the first indirect translation, for example, we not only quantified over sets—we quantified over a set of *everything*. But while this might work in certain contexts, it will not work in others. Maybe there are contexts in which we want to deny that there are sets altogether. Or maybe there are contexts in which we are happy to accept that there are sets, but want to maintain that sets are wellfounded (and so deny that there is a set of everything). The advantage of direct translations is that we can remain neutral on these questions. There is no need to take any particular position on the nature of sets in order to accept that there could have been something that does not actually exist.

The second case is similar. Maybe there are contexts in which we can compare things across world by quantifying over things like heights. But this strategy also comes with certain additional commitments, and so will not work in all cases. For example, you might want to use comparisons across worlds to do science without numbers. But if the only way to express such comparison is by quantifying over things like masses and charges and distances, then no such thing is possible, since quantifying over masses and charges and distances means quantifying over abstract things, when quantifying over abstract things is precisely what we are trying to avoid. Direct translations, on the other hand, come with no such commitments, so represent a powerful tool for simplifying the metaphysical commitments of science. This is just the point we considered at length in chapter one.

Yet another advantage of direct translations is that they are completely systematic. So long as a sentence of a regimented language has the right form, there will be a translation into an appropriate multi-dimensional language. Indirect translations, on the other hand, are *ad hoc* by their very nature. By way of illustration, consider a temporal case. Using a temporal regimented language, we can make the following pair of claims:

$$\exists t(Pt \wedge \exists s(t > s \wedge T(otws)))$$

$$\exists t(Pt \wedge \exists s(t > s \wedge A(otws)))$$

The first says that Obama at the present time is taller than Washington at some time in the past. And the second says that Obama at the present time *admires* Washington at some time in the past. Using a temporal analogue of Q_1, we can perhaps express the first by quantifying over heights. But even if so, the same strategy will not work for the second sentence—and this despite the fact that they have exactly the same form. Quantification over things like heights, then, is at best a stopgap measure. It may help us express comparative relations across time, but is of no help whatsoever when it comes to *non*-comparative relations.

2. Propositional Languages

Now that we have some motivation for multi-dimensional languages, we can get to work actually building them. We are going to start by building a hierarchy of propositional languages in this section. Then we are going to build a corresponding hierarchy of multi-dimensional quantified languages in section 4. Along the way, we will also generalize the familiar notion of a Kripke model from one dimension to arbitrarily many.

Each n-dimensional modal language $\mathcal{P}_n$ extends the language of propositional logic with certain sentential operators. In particular, the basic syntax of each $\mathcal{P}_n$ includes:

Atomic Sentences:	$P, Q, R\ldots$
Extensional operators:	$\wedge, \neg$
Intensional operators:	$\Box$ when $n \geq 1$
	$\uparrow$ when $n \geq 2$
	$\otimes_i$ for each $n \geq i \geq 2$
	Id when $n \geq 2$

The $\Box$, $\uparrow$, and $\otimes_i$ operators are one-place. The Id operator is zero-place, making it a kind of logical atomic sentence. The sentences of each $\mathcal{P}_n$ are then defined recursively in the way you would expect.[7] This gives us not just one, but an entire hierarchy of propositional modal languages. The most familiar are the language of propositional logic (which is $\mathcal{P}_0$) and the language of standard one-dimensional modal logic (which is $\mathcal{P}_1$). The rest of the languages are all multi-dimensional. Of those, the simplest is the two-dimensional language $\mathcal{P}_2$. As you can see from the above, that language extends propositional logic with exactly

7. Every atomic sentence is a sentence. Moreover, if ϕ and ψ are sentences, then $\neg\phi$ and $\phi \wedge \psi$ and $\Box\phi$ and $\uparrow\phi$ and $\otimes_i\phi$ and Id are sentences. These are all the sentences, at least when n is finite. When n is infinite, we may also want to add infinitary formation rules that allow for things like infinite conjunctions and infinite strings of operators.

four operators, which are $\Box$, $\uparrow$, $\otimes$, and Id.[8]

At this point, there are different sorts of models we could use. To start, we are going to use the simplest, which are multi-dimensional **Leibniz models**. These are structures of the form:

$$\mathcal{M}_n = \langle W, [\![\cdot]\!] \rangle$$

The first element is a non-empty set of possible worlds. The second is a valuation function. The subscripted n indicates the dimensionality of the modal. Each language $\mathcal{P}_n$ is then modeled by the class of models with the corresponding number of dimensions. So $\mathcal{P}_1$ is modeled using the class of one-dimensional Leibniz models, $\mathcal{P}_2$ is modeled with the class of two-dimensional Leibniz models, and so on.

There are two things that make a Leibniz model multi-dimensional instead of just one-dimensional. The first is the valuation function. Every n-dimensional model comes with an n-dimensional valuation function, which is a function that assigns each atomic sentence to a set of sequences of the appropriate length.[9] So:

$$[\![P]\!] \subset \mathbb{P}(W^n)$$

The second thing is that the central notion of truth, which is truth at a *sequence* of worlds rather than just truth at a world. This is defined recursively:

8. $\mathcal{P}_2$ has only one $\otimes_i$ operator, so we will generally drop the subscript, writing $\otimes$ instead of $\otimes_2$.

9. Going forward, when I talk about sequences, I mean sequences of worlds of length n, where n is the dimensionality of the corresponding model.

$$w_1w_2w_3... \models P \quad \text{iff} \quad w_1w_2w_3... \in [\![P]\!]$$

$$w_1w_2w_3... \models \neg\phi \quad \text{iff} \quad w_1w_2w_3... \not\models \phi$$

$$w_1w_2w_3... \models \phi \wedge \psi \quad \text{iff} \quad w_1w_2w_3... \models \phi \text{ and } w_1w_2w_3... \models \psi$$

$$w_1w_2w_3... \models \Box\phi \quad \text{iff} \quad uw_1w_2... \models \phi \text{ for every } u \in W$$

$$w_1w_2w_3... \models \uparrow\phi \quad \text{iff} \quad w_1w_1w_3... \models \phi$$

$$w_1...w_i... \models \otimes_i\phi \quad \text{iff} \quad w_i...w_1... \models \phi$$

$$w_1w_2w_3... \models \mathsf{Id} \quad \text{iff} \quad w_1 = w_2$$

As you can see, each operator corresponds to natural operation on sequences. The **box** operator $\Box$ pushes all the worlds in a sequence to the right, then puts a new world in the first position. When n is finite, the last world from the original sequence is deleted. The **up** operator $\uparrow$ copies the world from the first position into the second position. Each **swap** operator $\otimes_i$ swaps the world in the first position with the world in the i-th position. And finally, the **identity** operator Id checks to see if the first and second worlds are identical. A sentence is then true at a world w when true at the sequence $www...$ in which that world occupies every position. And a sentence is true in a model when true at every world.

Notice that on the above way of thinking about things, we get the familiar one-dimensional Leibniz models as a special case. In particular, when $n = 1$, Leibniz models consist of a set of worlds and a valuation function assigning each atomic sentences to a set of worlds.[10] Truth at a world is then defined recursively in the usual way:

10. Technically, the valuation assigns atomic sentences to sets of sequences of length one. But the sequences of length one are obviously isomorphic to the worlds. So we can, for practical purposes, think of the valuation function as assigning atomic sentences to sets of worlds.

$$w \models P \quad \text{iff} \quad w \in [\![P]\!]$$

$$w \models \neg\phi \quad \text{iff} \quad w \not\models \phi$$

$$w \models \phi \wedge \psi \quad \text{iff} \quad w \models \phi \text{ and } w \models \psi$$

$$w \models \Box\phi \quad \text{iff} \quad u \models \phi \text{ for every } u \in W$$

A sentence is then true in a model when true at every world.

Since the general case is a little abstract, it can be helpful to also consider the two-dimensional case. When $n = 2$, Leibniz models consists of a set of worlds and a valuation function. This valuation function assigns each atomic sentence to a set of *pairs* instead of just a set of worlds. The central notion of truth is then truth at a pair, which is defined recursively:

$$wv \models P \quad \text{iff} \quad wv \in [\![P]\!]$$

$$wv \models \neg\phi \quad \text{iff} \quad wv \not\models \phi$$

$$wv \models \phi \wedge \psi \quad \text{iff} \quad wv \models \phi \text{ and } wv \models \psi$$

$$wv \models \Box\phi \quad \text{iff} \quad uw \models \phi \text{ for every } u \in W$$

$$wv \models \uparrow\phi \quad \text{iff} \quad ww \models \phi$$

$$wv \models \otimes\phi \quad \text{iff} \quad vw \models \phi$$

$$wv \models \mathsf{Id} \quad \text{iff} \quad w = v$$

Truth at a pair is a purely technical notion but, for heuristic purposes, you might think of it in any of the following ways:

ϕ is true at wv iff ϕ is true at w from the perspective of v

ϕ is true at wv iff ϕ is true at w relative to v

ϕ is true at wv iff ϕ is true at w on the assumption that v is actual

As noted earlier, each of our operators corresponds to a certain natural operations on pairs. The box operator pushes the first world into the second position, then fills the first position with an arbitrary world. The up operators copies the world from the first position into the second position. The swap operator exchanges the two worlds. And the identity operator checks to see if the two worlds are identical. A sentence is then true at a world w when true at the pair ww. And a sentence is true in a model when true at every world.

Now that we have our basic languages and models, it will be helpful to also have certain non-basic expressions. These include:

$\Diamond\phi$	iff	$\neg\Box\neg\phi$
$\phi \vee \psi$	iff	$\neg(\neg\phi \wedge \neg\psi)$
$\phi \supset \psi$ ‘	iff	$\neg(\phi \wedge \neg\psi)$
$\phi \equiv \psi$	iff	$(\phi \supset \psi) \wedge (\psi \supset \phi)$
$\downarrow\phi$	iff	$\otimes\uparrow\phi$

This gives all of our languages a possibility operator and the usual range of truth-functional connectives. They also get a second copy operator ↓ called **down**. Where up copies the world from the first position into the second position, down does the reverse. It copes the world from the second position into the first position.[11]

11. When $n > 2$, there are other sorts of operators we might want to define. For example, we might want to define a family of operators $\otimes_{ij}$ that swap worlds between arbitrary coordinates. Or we might want to define a family of operators $\uparrow_{ij}$ that copy worlds between arbitrary coordinates. Or we might want to define a family of operators Id_{ij} that test for identity between arbitrary coordinates. You get the idea. Working out how to define these operators is an interesting exercise.

By this point, observant readers may have noticed that our basic operators are somewhat redundant. This because the up operator can be defined in terms of box and identity, given the following biconditional:

$$\uparrow\phi \equiv \Box(\mathsf{Id} \supset \phi)$$

This would seem to indicate that the up operator should simply be dropped from the list of basic operators. There are two reasons, though, that the up operator has been included. The first is that you might want to drop the identity operator from the language. You might, for example, want to deny that there is any sense to the question of whether or not there are distinct worlds that are also qualitative duplicates. Second, once we have a more general approach to models, the above biconditional can fail. So it is not true *in general* that up can be defined in terms of box and identity, even though this happens to be true when using Leibniz models.[12]

3. Multi-Dimensional Kripke Models

Leibniz models encode a certain view about the nature of possibility, one on which the range of possibilities is fixed as a matter of necessity. Whatever could have been possible is already possible. Whatever could have been necessary is already necessary. Objective modalities, like metaphysical necessity and physical necessity, are generally thought to satisfy these conditions, so this is perhaps not a decisive problem. But you might also find yourself in

12. More precisely: Once we get to Kripke models, up can be defined in terms of box and identity iff the accessibility relation is reflexive.

the minority, and so convinced that the range of possibilities is contingent. It would be nice, then, if we had models that let us make sense of this idea.

Even if you think that the range of objective possibilities is necessary, still, you might want to use modal logic to reason about other intensional notions like knowledge and permissibility. But in those cases, the assumptions embedded in Leibniz models will clearly fail. We do not always know what we know, nor are the actions we perform always permissible. But in order to make sense of the first idea, we will need to deny S4 and, in order to make sense of the second, we will need to deny T. Since Leibniz models validate both, this means that we will need a more general way to think about models. What's more, the reasons to adopt a multi-dimensional framework generally carry over from the modal case to these other sorts of notions. Just as you might want to express the idea that faculty meetings *could* have been less boring than they are, you might want to express the idea that they *should* be less boring than they are, or that you *thought* they would be less boring than they are. And these sorts of claims, as we have already seen, are naturally expressed using a multi-dimensional framework.

The natural way to generalize Leibniz models is by using multi-dimensional **Kripke models**. These are just like Leibniz models, but with the addition of an accessibility relation $R \subset W^2$. This makes them structures of the form:

$$\mathcal{M}_n = \langle W, R, [\![\cdot]\!] \rangle$$

Truth at a sequence is defined the same way as before, with the exception of the clause for the box operator:

$$w_1 w_2 w_3 ... \models \Box\phi \quad \text{iff} \quad u...w_i...w_n \models \phi \text{ for every } u \in W \text{ such that } uw_1 \in R$$

In the one-dimensional case, this gives us:

$$w \models \Box\phi \quad \text{iff} \quad u \models \phi \text{ for every } u \in W \text{ such that } uw \in R$$

And in the two-dimensional case, we have:

$$wv \models \Box\phi \quad \text{iff} \quad uw \models \phi \text{ for every } u \in W \text{ such that } uw \in R$$

A sentence is then true at a world w when true at the sequence www... in which that world occupies every position. And it is true in a model when true at every world.[13]

To my knowledge, multi-dimensional Kripke models appear nowhere in the literature. This is surprising, given their relative simplicity and flexibility. For whatever reason, the general focus has been on **concrete models**.[14] In the two-dimensional case, these are structures:

$$\mathcal{M}_2 = \langle S, [\![\cdot]\!] \rangle$$

The first element is a set of possible pairs. The second is a valuation function assigning each atomic sentence to a set of possible pairs, meaning that $[\![P]\!] \subset S$. Leibniz models are then

13. An alternative approach would be to say that n dimensional models always have an $n+1$ place accessibility relation instead of an n place accessibility relation. So for example, in the two-dimensional case, we would use a *three*-place accessibility relation instead of a two-place relation. $\Box\phi$ would then be true at a pair wv iff there was a u such that $Ruwv$. This would let us make sense of the idea that whether or not a certain world w is accessible from v is itself a perspectival matter. From one world x, w is accessible from v. But from another world y, w is not accessible from v. Whether are any useful applications for this extra degree of relativity is an interesting question.

14. See Marx and Venema (1997) for more on this.

the special case in which $P = W^2$. But while all Leibniz models are concrete models, not all concrete models are Leibniz models, since the set of possible pairs could also be a strict subset of W^2 meeting certain closure conditions. Those closure conditions are that $ww \in P$ and $vw \in P$ whenever $wv \in P$. These are needed to ensure that the up operator and swap operator are defined. A sentence is then true in a model when true at all possible pairs.

The problem with concrete models is that the set of possible pairs is in effect forced to play the role of both a set of worlds and an accessibility relation. You can see this by looking at the clause for the necessity operator, which is going to be something like:

$$wv \models \Box\phi \quad \text{iff} \quad uw \models \phi \text{ for every } uw \in S$$

This has certain unfortunate consequences. For example, sentences can only be true at a pair if that pair is in the set of possible pairs. So in order to make a sentence ϕ true at a world w, we have to include ww in the set of possible pairs. But once ww is in the set of possible pairs and ϕ is true at ww, it follows that $\Diamond\phi$. So concrete models validate the T axiom. Moreover, they also validate a two-dimensional version of the B axiom, since the closure conditions force the set of possible pairs to be symmetric. We could try to fix the problem by weakening the closure conditions but, in that case, the swap operator will not always be defined.

Multi-dimensional Kripke models can avoid these problems because they have both a set of worlds and an accessibility relation, just like in the one-dimensional case. This gives them the flexibility we need to model weaker systems.

4. Quantified Languages

We built a hierarchy of propositional modal languages $\mathcal{P}_n$ in section 2. We are now going to build a corresponding hierarchy of quantified modal languages $\mathcal{Q}_n$, each of which have the following basic syntax:

Non-logical predicates:	$P, Q, R...$
Logical predicates:	$=$
Variables:	$x, y, z...$
Names:	$c, d, e...$
Quantifiers:	$\forall$
Extensional operators:	$\wedge, \neg$
Intensional operators:	$\Box$ when $n \geq 1$
	$\downarrow$ when $n \geq 2$
	$\otimes_i$ for each $n \geq i \geq 2$
	Id when $n \geq 2$

Every predicate is assigned a fixed finite arity and the sentences of each language are defined recursively in the usual way.[15] The non-basic expressions then include the ones from earlier, along with:

15. Let $t_1, ..., t_n$ be terms and P^n an arbitrary n-place predicate. Then $P^n(t_1, ..., t_n)$ and $(t_1 = t_2)$ are sentences. And if ϕ and ψ are sentences, then so are $\forall x(\phi)$, $(\phi \supset \psi)$, $\neg(\phi)$, $\Box(\phi)$, $\downarrow(\phi)$, $\otimes_i(\phi)$, and Id. Nothing else is a sentence. As in the propositional case, we may want to use infinitary formation rules when the number of dimensions is infinite. The assumption of fixed finite arities could also be easily dropped.

$\exists x\phi \quad \text{iff} \quad \neg\forall x\neg\phi.$

$Et \quad \text{iff} \quad \exists x(t = x)$

This gives each of our languages both an existential quantifier and an existence predicate.

Once we have our languages, they can each be modeled using the corresponding class of quantified Kripke models. These are just like the propositional Kripke models from the last section, but have the extra structure needed to handle things like quantifiers and predicates.[16]

$$\mathcal{M}_n = \langle W, R, D, d, [\![\cdot]\!]\rangle$$

The first three elements are a domain of worlds, an accessibility relation, and a domain of possible individuals. The fourth is a domain function mapping each sequence of worlds to a set of possible individuals:

$$d : W^n \longrightarrow \mathbb{P}(D)$$

Finally, we have an n-dimensional valuation function, which does two things:

$$[\![c]\!] \in D$$

$$[\![P]\!] : W^n \longrightarrow D^m$$

First, it assigns every name to a possible individual. And second, it assigns each m-place predicate to an n-dimensional intension, which is just a function from sequences of worlds (of length n) to sets of sequences of individuals (of length m). The central notion of truth is

16. Generally speaking, we will refer to both propositional Kripke models and quantified Kripke models as just Kripke models, letting context determine which sort of models we have in mind.

then truth at a sequence *relative* to a variable assignment. This is defined recursively, with the clauses for the sentential operators being the same as before. The new clauses are:

$$w_1w_2w_3... \models_\sigma P(t_1,...,t_n) \quad \text{iff} \quad \sigma(t_1)...\sigma(t_m) \in [\![P]\!](w_1w_2w_3...)$$

$$w_1w_2w_3... \models_\sigma t_1{=}t_2 \quad \text{iff} \quad \sigma(t_1){=}\sigma(t_2)$$

$$w_1w_2w_3... \models_\sigma \forall x\phi(x) \quad \text{iff} \quad \text{every } \tau \text{ is such that } w_1w_2w_3... \models_\tau \phi(x)$$

A sentence is true at a sequence when true at that sequence relative to all variable assignments. A sentence is true at a world when true at the sequences in which that world occupies every position. And a sentence is true in a model when true at every world.[17]

As before, this gives us familiar one-dimensional Kripke models as the special case in which $n = 1$. The two-dimensional case in which $n = 2$ is then as follows: First we have a set of worlds, an accessibility relation, and a domain of possible individuals. This is all the same as in the one-dimensional case. What makes two-dimensional models different is then two things. First, the domain function maps *pairs* of worlds to sets of individuals. And second, the valuation function maps every predicate to a function mapping *pairs* of worlds to sets of sequences of individuals of the appropriate length.

What does this all mean exactly? You might think of $a \in d(wv)$ as encoding the fact that a exists at w from the perspective of v. Our two-dimensional models thus allow for a kind of modal ontological relativity, since we can have $a \in d(wv)$ but not $a \in d(wu)$. The

17. A variable assignment σ is a function mapping each term in the language to a member of D. In the case of names, it does so in a way that matches the valuation function. So $\sigma(c) = [\![c]\!]$. In the above recursive clauses, the variable assignment τ is just like σ, with the possible exception that $\tau(x) \neq \sigma(x)$ and the added requirement that $\tau(x) \in d(w_1w_2w_3...)$.

individual a may exist at w from the perspective of v, but not exist at w from the perspective of some other world u.[18] Once we get to the valuation function, one-place predicates are mapped to functions from pairs of worlds to sets of individuals. Two-place predicates are mapped to a function from pairs of worlds to sets of pairs of individuals. And so on. How you should think about these functions depends somewhat on the intended reading of the relevant predicates. But for example, if W denotes the *being filled with water* property, you might think of $a \in [\![W]\!](wv)$ as indicating that a has the property of being filled with water at w from the perspective of v. And if T denotes the *taller than* relation, you might think of $ab \in [\![T]\!](wv)$ as indicating that a at w is taller than b at v.

Once we get to truth in a model, the important notion is truth at a pair relative to a variable assignment. This is defined recursively, with the clauses for the sentential operators being the same as before. The new clauses are:

$$wv \models_\sigma P(t_1, ..., t_m) \quad \text{iff} \quad \sigma(t_1)...\sigma(t_n) \in [\![P]\!](wv)$$

$$wv \models_\sigma t_1{=}t_2 \quad \text{iff} \quad \sigma(t_1){=}\sigma(t_2)$$

$$wv \models_\sigma \forall x\phi(x) \quad \text{iff} \quad \text{every } \tau \text{ is such that } wv \models_\tau \phi(x)$$

A sentence is true at ww when true at ww relative to all variable assignments σ. A sentence is true at a world w when true at the pair ww. And a sentence is true in a model when true at all worlds.

18. When this relativity is unwanted, we can just use the class of frames with non-relative domains. That is, we can just use those frames in which $d(wv) = d(wu)$ for all worlds.

5. Proof Theory for PML

Proof systems for quantified modal logic are naturally built in stages. First, we build an appropriate propositional modal logic. Then we add the rules and axioms of an appropriate quantified predicate logic. Finally, we glue everything together by adding certain further principles as needed. In what follows, we are going to use this process to build various two-dimensional quantified modal logics. We are focusing on the two-dimensional case for simplicity and concreteness. But the basic approach extends to higher dimensions as well.

The propositional modal logic $\mathbf{K} = \mathbf{K_1}$ is the foundation of one-dimensional quantified modal logic. The first step towards building a multi-dimensional QML, then, is to construct an appropriate multi-dimensional analogue. Since our ultimate goal is to build a two-dimensional QML, that means building $\mathbf{K_2}$.

The principles of $\mathbf{K_2}$ fall into two broad categories. The first category consists of those principles that are straightforward analogues of the rules and axioms of $\mathbf{K_1}$.

MP	$\phi \supset \psi, \phi \Rightarrow \psi$
RN	$\phi \Rightarrow \Box\phi$
	$\phi \Rightarrow \uparrow\phi$
	$\phi \Rightarrow \otimes\phi$
K	$\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$
	$\uparrow(\phi \supset \psi) \supset (\uparrow\phi \supset \uparrow\psi)$
	$\otimes(\phi \supset \psi) \supset (\otimes\phi \supset \otimes\psi)$
PL	ϕ when ϕ is a theorem of propositional logic

As you can see, the system has two basic rules, which are modus ponens and the rule of necessitation. The axiom K then says that all three operators distribute over material conditionals. And PL says that the theorems of classical propositional logic are also theorems of $\mathbf{K_2}$. These together ensure that all of the relevant operators are normal.

The second category of principles consists of axioms governing the interaction of our various operators. These are what give $\mathbf{K_2}$ its two-dimensional character.

A1 $\quad \uparrow\neg\phi \equiv \neg\uparrow\phi$

A2 $\quad \otimes\neg\phi \equiv \neg\otimes\phi$

A3 $\quad \uparrow\otimes\phi \equiv \uparrow\phi$

A4 $\quad \otimes\otimes\phi \equiv \phi$

A5 $\quad \uparrow\phi \supset \Box\otimes\uparrow\phi$

A6 $\quad \uparrow\Box\phi \equiv \Box\phi$

A7 $\quad \mathsf{Id} \supset (\uparrow\phi \equiv \phi)$

A8 $\quad \uparrow\mathsf{Id}$

A9 $\quad \mathsf{Id}$

Thinking in terms of pairs, A1 says that the result of copying is always unique. A2 says that the result of swapping is always unique. A3 says that copying and then swapping is the same as just copying. A4 says that swapping twice is the same as not swapping at all. A5 says that the box operator has no effect on the second coordinate. A6 says that accessibility is fully determined by the first coordinate. A7 says that when two worlds are identical, copying does nothing. A8 says that copying always results in identical worlds. And finally, A9 says that our point of departure in modal space is always a *world*, rather than just an arbitrary pair. Note that in order to avoid triviality, the rule of necessitation needs to be restricted to exclude this last axiom.[19] If the identity operator is dropped from the language, then the last three axioms are replaced with the following two:

A7* $\uparrow(\uparrow\phi \equiv \phi)$

A8* $\uparrow\phi \equiv \phi$

The rule of necessitation is then restricted to exclude A8* instead of A9.

Once we have $\mathbf{K_2}$, stronger systems can be build by adding various characteristic axioms. This process is familiar from the one-dimensional case, but the axioms themselves are somewhat different. Some of the most important are listed in the table on the next page. As you can see, all of the usual one-dimensional axioms have two-dimensional analogues. An axiom that will play an especially prominent role later on is TB, which is so-called because it is equivalent to the conjunction of T and B.

	1D Axiom	**2D Axiom**	**Frames**
D	$\Box\phi \supset \Diamond\phi$	$\Box\phi \supset \Diamond\phi$	serial
T	$\Box\phi \supset \phi$	$\Box\phi \supset \uparrow\phi$	reflexive
TB		$\Box\phi \supset \otimes\phi$	reflexive and symmetric
B	$\Diamond\Box\phi \supset \phi$	$\Diamond\Box\phi \supset \Diamond\otimes\phi$	symmetric
S4	$\Box\phi \supset \Box\Box\phi$	$\Box\uparrow\phi \supset \Box\Box\uparrow\phi$	transitive
S5	$\Diamond\Box\phi \supset \Box\phi$	$\Diamond\Box\phi \supset \Box\Box\otimes\phi$	euclidean

As far as I can tell, there is no (non-conjunctive) analogue in the one-dimensional case, so that space in the left column has been left blank.

19. That is, RN says that $\Box\phi$, $\downarrow\phi$, and $\otimes$ are theorems when ϕ can be proved without A9.

6. Proof Theory for QPL

Once we have a basic two-dimensional PML, the next step towards building a two-dimensional QML is identifying an appropriate quantified predicate logic. For our purposes, that quantified predicate logic is going to be the free logic **F**.

UG $\phi \Rightarrow \forall x\phi$

B1 $\forall x\phi x \supset (Et \supset \phi t)$

B2 $\forall x(\phi \supset \psi) \supset (\forall x\phi \supset \forall x\psi)$

B3 $\phi \supset \forall x\phi$, when x is not free in ϕ

B4 $t = t$

B5 $t = s \supset (\phi \supset \phi[t/s])$

F $\forall xEx$

First, we have the rule of universal generalization. There are then various axioms governing the universal quantifier and various axioms governing identity.[20] We then have the axiom F, which is so-called because it is what distinguished **F** from classical predicate logic, which has the stronger axiom:

C Et

When combined with the rule of necessitation, classical predicate logic entails that everything exists necessarily. But in certain contexts, this is a result we would like to

20. For axiom B5, $\phi[t/s]$ is a sentence that result from ϕ by replacing any number of instances of s with t.

avoid. So we are using the free logic **F** instead.[21]

7. Proof Theory for QML

You might have hoped that we could build a sensible quantified modal logic just by adding **F** to $\mathbf{K_n}$. But in fact, this is not the case. The combined logic $\mathbf{FK_1}$ is not complete with respect to *any* class of Kripke frames. And the same goes for $\mathbf{FK_2}$. What this means is that in order to build a complete system, further principles need to be added.[22]

Spelling this out a bit, a QML is complete only if it passes two important tests. The first is whether the system can prove certain modal principles involving identity. These include the **preservation of identity** and the **preservation of distinctness**.

PI $\quad t = s \supset \Box(t = s)$

$t = s \supset \uparrow(t = s)$

$t = s \supset \otimes(t = s)$

PD $\quad t \neq s \supset \Box(t \neq s)$

$t \neq s \supset \uparrow(t \neq s)$

$t \neq s \supset \otimes(t \neq s)$

The preservation of identity can be proved using the fact that all of the relevant operators are normal. The situation is somewhat different, though, when we get to the preservation

21. Note that besides the above, **F** also includes MP and PL, which are not listed here, since they were listed earlier.

22. Going forward, when we say that a QML is complete, we mean complete with respect to some class of Kripke models.

of distinctness. $PD_\otimes$ can be proved using $\mathbf{FK_2}$. We can also show that $PD_\uparrow$ entails $PD_\Box$. The problem is that $\mathbf{FK_2}$ and $PD_\uparrow$ are independent. So there is no way to prove the full preservation of distinctness in $\mathbf{FK_2}$.[23]

The second test is whether a system allows us to generalize in a sufficiently wide range of contexts. In ordinary quantified predicate logic, for example, we can generalize in the scope of material conditionals. This point might be put more precisely by saying that the rule of **conditional universal generalization** is valid:

CUG $\quad (\phi \supset \psi) \Rightarrow (\phi \supset \forall x\psi)$ when x is not free in $(\phi \supset \forall x\psi)$

Some systems treat this as a basic rule. Others treat it as derived. Once we move to quantified modal logic, a system is complete only if we can do the same with **strict conditionals**. These are sentences $\phi \multimap \psi$ of the form

$$\phi_1 \supset (\phi_2 \supset \dots (\phi_n \supset \psi) \dots)$$

where any of the embedded material conditionals are prefaced by any number of intensional operators. So for example, all of the following count as strict conditionals:

$$\Box(\phi \supset \psi)$$

$$\otimes(\phi_1 \supset (\phi_2 \supset \psi))$$

$$\phi_1 \supset \Box(\phi_2 \supset \otimes\downarrow(\phi_3 \supset \psi))$$

What we can show is that a system is complete only if it validates **modal universal generalization**, which says precisely that we can generalize in the scope of strict conditionals.

23. The one-dimensional results described in this section are all well-known. The two-dimensional results will be proved later on.

MUG $(\phi \dashv\exists\, \psi) \Rightarrow (\phi \dashv\exists\, \forall x\psi)$ when x is not free in $(\phi \dashv\exists\, \forall x\psi)$

The problem is that systems like $\mathbf{FK_1}$ and $\mathbf{FK_2}$ do not validate MUG. And so it follows that the systems are not complete.

There are two main approaches to fixing the problem. The first is to take certain mixed principles—like MUG or PD—as basic. These principles are "mixed" in the sense that they directly govern the interaction of modal ideology and quantificational ideology. Using this strategy, we can build both $\mathbf{Q_1}$ and $\mathbf{Q_2}$. The first is the logic of all one-dimensional Kripke models, and the second is the logic of all two-dimensional Kripke models.

System	**Components**	**Frames**
$\mathbf{Q_1}$	$\mathbf{FK_1}$, MUG, $PD_\Box$	all
$\mathbf{Q_2}$	$\mathbf{FK_2}$, MUG, $PD_\uparrow$	all
$\mathbf{QB_1}$	$\mathbf{FK_1}$, B	symmetric
$\mathbf{QTB_2}$	$\mathbf{FK_2}$, TB	reflexive and symmetric

The other strategy is to strengthen the underlying propositional modal logic. In the one-dimensional case, we can derive both MUG and PD in any system that has the B axiom. This means that, for example, the system $\mathbf{QB_1}$ is complete with the respect to the class of all symmetric Kripke frames. Later, we are going to show that in the two-dimensional case, we can derive MUG and PD in any system that has TB. This means that the system $\mathbf{QTB_2}$, for example, is complete with respect to the class of reflexive and symmetric Kripke frames.

Both of these strategies strike me as completely sensible. The advantage of the first is flexibility. If you are doing quantified epistemic logic, say, you will probably want to deny that if you might know a proposition, then that proposition is true. But in that case, the B axiom will fail, so the first strategy is the only game in town. The advantage of the second strategy is simplicity. If you are in a context where you have B axiom (in the one-dimensional case) or TB axiom (in the two-dimensional case), you might as well use them to derive principles like MUG and PD. The main point for our purposes is that *both* strategies are still available after moving from one-dimensional QML to two-dimensional QML.

8. Barcan Formulas

Once we have a complete system, stronger systems can be built by adding characteristic axioms. We saw some of these when we axiomatized two-dimensional PML in section 5. Once we move to a two-dimensional QML, the main new possibilities are axioms governing the interaction of operators and quantifiers.

	Axiom	**Frames**
BF	$\forall x \Box \phi \supset \Box \forall x \phi$	$\Box$-decreasing
	$\forall x \uparrow \phi \supset \uparrow \forall x \phi$	$\uparrow$-decreasing
CBF	$\Box \forall x \phi \supset \forall x \Box \phi$	$\Box$-increasing
	$\uparrow \forall x \phi \supset \forall x \uparrow \phi$	$\uparrow$-increasing

BF and CBF are familiar in the case of the necessity operator. The first tells us that everything that could have existed already exists. And the second says that everything that exists also exists necessarily. Putting them together, we get the claim that **ontology is fixed**. The things that exist do not change as we switch from considering one possible world to considering another. BF and CBF are somewhat less familiar in the case of the up operator. The first says that whatever exists at a world from its own perspective also exists at that world from the perspective of any other world. The second says that whatever exists at a world from the perspective of any world exists at that world from its own perspective. Putting these together, we get the idea that **ontology is non-relative**. What exists at a world does not depend on which world is actual.

Putting the issue more precisely, once we have a two-dimensional QML, we can distinguish two senses in which a frame might be increasing or decreasing.

$\Box$-increasing	$d(wv) \subset d(uw)$ when Ruw
$\uparrow$-increasing	$d(wv) \subset d(ww)$
$\Box$-decreasing	$d(uw) \subset d(wv)$ when Ruw
$\uparrow$-decreasing	$d(ww) \subset d(wv)$

Whether a domain is $\Box$-increasing or $\Box$-decreasing is a matter of whether the domain always grows or shrinks as we consider *different* possible worlds from the perspective of the *same* world. Whether a domain is $\uparrow$-increasing or $\uparrow$-decreasing, on the other hand, is a matter of whether the domain grows or shrinks as we consider the *same* possible world from the perspective of *different* worlds. Having these two senses in which a domain might be increasing or decreasing thus raises the possibility of holding certain novel—and for the most part unexplored—positions in modal ontology.[24]

24. For example, you might hold that ontology is both fixed and relative. On this view, from the perspective of any world v, the same things exist at every possible world w, and so ontology is a completely necessary matter from the perspective of every world. But nevertheless, it may be that *different things* exist at every world from the perspective of v than exist at every world from the perspective of some other world u. Whether

9. PML Completeness

We now know how to build a variety of two-dimensional system. Next, we are going to show that they are complete. First, we are going to show that the propositional modal logic $\mathbf{K_2}$ is complete. Then we are going to show that the quantified modal logics $\mathbf{Q_2}$ and $\mathbf{QTB_2}$ are complete.

The most familiar strategy for proving completeness is to build a canonical model. A canonical model is one in which the domain of worlds is the set of all maximal consistent sets of sentences. The valuation function then assigns each atomic sentences to the set of worlds that have that sentence as a member. And the accessibility relation is similarly fixed by the facts about which sentences are members of which sets.[25]

The problem with the canonical approach is there is no straightforward way of extending to two or more dimensions. In the two-dimensional case we can, of course, form the set of all maximal consistent sets of sentences, and so we might try using that set as our domain of worlds. But then how are we going to build the valuation function? In the one-dimensional case, we can just stipulate that a sentence is true at a world iff it is a member of that world. But the equivalent move in the two-dimensional case—saying that a sentence is true at a pair iff is a member of that pair—makes no sense. Sentences are

this is a view you should hold is a substantive question in metaphysics, so a matter best left for another time. For a defense of this sort of relative necessitism, see (Murray and Wilson 2012). For my own part, I am inclined toward relative contingentism, so am inclined to deny BF and CBF in both forms.

25. For those unfamiliar with canonical models, see chapter six of (Hughes and Cresswell 1996).

not members of pairs of worlds. We could try identifying truth at a pair with, say, being a member of the world in the first coordinate. But this only works if the same sentences are always true at pairs *wv* and *wu*, which is not generally the case. Other ideas along these lines will fair no better, and for similar reasons.

The solution is to use what Blackburn, Rijke, and Venema (2010) call the **step-by-step approach**. The step-by-step approach, unlike the canonical approach, builds models by adding worlds one at a time. In the one-dimensional case, we might start by adding a single world. We then assign that world to a maximal consistent set of sentences. Unlike the canonical approach, then, we are not *identifying* worlds with maximal consistent sets. We are associating them. At that point, what we have is *almost* a model, but will almost certainly have certain gaps. So we make a list of the gaps and fill them in by adding more worlds one-by-one. Once all the gaps are filled, we have a model, and so have proved completeness. As you will see in what follows, this basic process can be naturally extending to two dimensions and beyond.

Going forward, we will need to distinguish between two senses in which a set of sentences can be consists. We will say that a set is **strongly consistent** when no contradiction can be derived in $\mathbf{K_2}$. And we will say that a set is **weakly consistent** when no contradiction can be derived in $\mathbf{K_2}$ without using A9. Generally speaking, when we talk about a set being consistent full stop, what we mean is that it is weakly consistent. To simplify matters, we will also assume throughout that all of our languages are countable, though this is also an assumption that could be easily dropped. Without further ado then, let's get started.

Definition 9.1. *A **network** is a tuple $\langle W, R, f\rangle$ consisting of a set of worlds W, an accessibility relation $R \subset W^2$, and a partial function f assigning elements of W^2 to maximal consistent sets.*

Definition 9.2. *A network is **coherent** when:*

(C1) $\otimes\phi \in f(wv)$ *iff* $\phi \in f(vw)$

(C2) $\uparrow\phi \in f(wv)$ *iff* $\phi \in f(ww)$

(C3) $\mathsf{Id} \in f(wv)$ *iff* $w = v$

(C4) If $wv \in R$*, then* $f(wv)$ *is defined.*

(C5) If $\phi \in f(wv)$ *and* $uw \in R$ *then* $\Diamond\phi \in f(uw)$

Definition 9.3. *A network $\mathcal{N}$ is **saturated** when:*

If $\Diamond\phi \in f(wv)$*, then there is a* $uw \in R$ *such that* $\phi \in f(uw)$

Definition 9.4. *A network is **perfect** when it is both coherent and saturated.*

Definition 9.5. *A network $\mathcal{N}$ and a model $\mathcal{M}$ **correspond** if they have the same frame and*

$\mathcal{M}, wv \models \phi$ *iff* $\phi \in f(wv)$

whenever $f(wv)$ is defined.

Readers familiar with the step-by-step approach may have noticed that we are using *partial* networks, which is somewhat non-standard. That is, we are allowing for the idea that a network may contain pairs of worlds wv such that $f(wv)$ is not defined. Moreover, not only are we allowing networks to be partial, we are even allowing *perfect* networks to be partial. This means, in effect, that even a perfect network may not fully determine which sentences are true at which worlds.

Now, you might think that this should all lead to disaster. Surely the point of building a perfect network is to fix on a unique corresponding model? And so surely we need perfect networks to be *total* instead of partial? But here is why we are using partial networks—and why this can be expected to work. We said earlier that a sentence is true in a model iff it is true at all worlds. That is, it is true in a model iff it is true at all pairs ww with the same world in each coordinate. The **realistic** pairs are then those pairs wv such that either $w = v$ or Rwv or Rvw. What we could show, with a bit of work, is that truth in a model is full determined by the realistic pairs. The unrealistic pairs are irrelevant. What a perfect network does, then, is assign a maximal consistent set to every *realistic* pair. Some of the unrealistic pairs may be left out. A perfect network can then have many corresponding models because there are many different ways of arbitrarily filling in truth at the undefined—and therefore unrealistic—pairs. In the next lemma, for example, we are going to construct a corresponding model by arbitrarily choosing to make all atomic sentences false at all undefined pairs. But other arbitrary choices would work just as well.

This explains why using partial networks can be expected to work. But why is it necessary? The use of partial networks is necessary because there can be cases in which no maximal consistent set can be assigned to an unrealistic pair without making the network incoherent. Using partial networks, then, gives us a simple way of sidestepping the problem.

Lemma 9.1 (Truth Lemma). *Every perfect network $\mathcal{N}$ has a corresponding model $\mathcal{M}$.*

Proof. Let $\mathcal{N}$ be a perfect network. We then construct a corresponding model $\mathcal{M}$ by using the same frame, with the valuation function set to

$$[\![\phi]\!] = \{\langle w, v\rangle \mid f(wv) \text{ is defined and } \phi \in f(wv)\}$$

for every atomic sentence ϕ. That the resulting model is in fact a *corresponding* model is then shown using a straightforward induction on the complexity of sentences. ■

Lemma 9.2. *Every strongly consistent set has a countable coherent network. In particular, given any any strongly consistent set S, there is a countable coherent network $\mathcal{N} = \langle W, R, f \rangle$ such that $S \subset f(wv)$ for some $w, v \in W$.*

Proof. Let S be any strongly consistent set. There is thus a maximal consistent set T extending S by Lindenbaum's lemma. There are then two cases. If $\mathsf{Id} \in T$, then we construct an $\mathcal{N}$ such that $W = \{w\}$ and $R = \emptyset$ and $f(ww) = T$. On the other hand, if $\mathsf{Id} \notin T$, we construct an $\mathcal{N}$ such that:

$$W = \{w, v\}$$

$$R = \emptyset$$

$$f(wv) = T$$

$$f(vw) = \{\phi \mid \otimes\phi \in T\}$$

$$f(ww) = \{\phi \mid \uparrow\phi \in T\}$$

$$f(vv) = \{\phi \mid \otimes\uparrow\phi \in T\}$$

Given the axioms, it can be easily verify that in either case, the result is a coherent network of the required kind. ■

We now know that every strongly consistent set of sentences has a coherent network. But for all we have said, that network may have certain defects that prevent it from being saturated. We are now going to show how to fix those defects, with the main step being the proof of something called the repair lemma.

Definition 9.6. *Let $\mathcal{N} = \langle W, R, f \rangle$ be any network. A* ***defect*** *is a triple $\langle w, v, \Diamond\phi \rangle$ such that $\Diamond\phi \in f(wv)$, but there is no $u \in W$ such that $\langle u, w \rangle \in R$ and $\phi \in f(uw)$.*

Definition 9.7. *A network $\mathcal{N}^*$* ***extends*** *the network $\mathcal{N}$ when $W \subset W^*$ and $R \subset R^*$ and $f(wv) = f^*(wv)$ whenever $f(wv)$ is defined.*

Lemma 9.3. *If S is consistent and $\Diamond\phi \in S$, then there is a maximal consistent set T extending $S^* = \{\psi \mid \Box\psi \in S\} \cup \{\phi\}$.*

Proof. Suppose that S is consistent, but S^* not consistent. So there are $\Box\psi_1, ..., \Box\psi_n \in S$ such that $\psi_1 \wedge ... \wedge \psi_n \supset \neg\phi$. We thus have $\Box\psi_1 \wedge ... \wedge \Box\psi_n \supset \Box\neg\phi$ by RN and K. But then $\Box\neg\phi \in S$, meaning that S is not consistent, which is contrary to assumption. So S^* is consistent. There is thus a maximal consistent set T extending S^* by Lindenbaum's lemma. ∎

Lemma 9.4 (Repair Lemma)**.** *For any defect of any countable coherent network $\mathcal{N}$, there is a countable coherent network $\mathcal{N}$ extending $\mathcal{N}^*$ without that defect.*

Proof. Let $\mathcal{N}$ be any coherent network with the flaw $\langle w, v, \Diamond\phi \rangle$. Furthermore, let $S = \{\psi \mid \Box\psi \in f(wv) \cup \{\phi\}$. We then check to see if there is already a $u \in W$ such that $S \subset f(uw)$. If so, then all we have to do is let $W^* = W$ and $R^* = R \cup \{\langle u, w \rangle\}$ and $f^* = f$. On the other hand, suppose that there is no such $u \in W$. In that case, we have to add one. So we choose any $u \notin W$ and let $W^* = W \cup \{u\}$ and $R^* = R \cup \{u, w\}$. We then let:

$$
\begin{aligned}
f(wv) &= \text{an arbitrary maximal consistent set } T \text{ extending } S \\
f(vw) &= \{\phi \mid \otimes\phi \in T\} \\
f(ww) &= \{\phi \mid \uparrow\phi \in T\} \\
f(vv) &= \{\phi \mid \otimes\uparrow\phi \in T\}
\end{aligned}
$$

That there is such a maximal consistent set follows from the previous lemma. Using the axioms, it can then be confirmed that in either case, we have a coherent network $\mathcal{N}^*$ extending $\mathcal{N}$, but without the original flaw. ■

Lemma 9.5. *Every countable coherent network $\mathcal{N}$ can be extended to a countable perfect network $\mathcal{N}^*$.*

Proof. Let $\mathcal{N}$ be a countable coherent network and W^* the result of adding countably many new worlds to W. The set of potential flaws is then $W^* \times W^* \times \mathsf{Sent}$, which is also countable. This means that we can form an enumeration of the potential flaws, which we then use to inductively define a chain of networks. For the base case, $\mathcal{N}_0 = N$. Now suppose that we have a network N_n and want to construct N_{n+1}. First, we check to see if the potential flaw $n+1$ is a flaw of N_n. If not, then we move on, letting $N_{n+1} = N_n$. If so, then we repair the flaw using the method from the repair lemma, drawing unused worlds from W^* as needed. The result of the repair is then N_{n+1}. We then claim that the union N^* of this chain of networks is a perfect network extending $\mathcal{N}$.

$\mathcal{N}^*$ clearly extends $\mathcal{N}$. That $\mathcal{N}^*$ is coherent follows from the fact that (a) the base case is coherent, (b) the inductive step preserves coherence, and (c) the taking of unions preserves coherence. To show that $\mathcal{N}^*$ is saturated, suppose for reductio that it has a certain flaw $\langle w, v, \Diamond\phi \rangle$. Every flaw is a potential flaw, so thus has a place in our ordering. This means that we can refer to the alleged flaw as n. But then by construction, there is a network $\mathcal{N}_n$ in our chain that does not have n as a flaw. And if $\mathcal{N}_n$ does not have n as a flaw, then neither does any network extending $\mathcal{N}_n$. So n is not a flaw of $\mathcal{N}^*$ either, contrary to assumption. This means that $\mathcal{N}^*$ has no flaws and is therefore perfect. ■

Theorem 9.1 (Completeness). *Every strongly* $\mathbf{K_2}$*-consistent set of sentences has a two-dimensional propositional Kripke model.*

Proof. Immediate from the preceding. ∎

Now that we have a completeness result for $\mathbf{K_2}$, you might also want similar results for stronger logics. For example, you might want to show that the logic $\mathbf{KTB_2}$ is complete with respect to the class of reflexive and symmetric two-dimensional Kripke frames. That proof is basically the same as the one we just gave. The only difference is that we would have to strengthen the definition of coherence to include the reflexivity and symmetry of the accessibility relation. At every stage in the process, we would then have to confirm that these additional coherence requirements were met.

10. QML Completeness

Now that we have a completeness result for $\mathbf{K_2}$, we would like to prove a similar result for $\mathbf{Q_2}$. This can be done using the **method of possible names**.[26] The basic idea is to replace maximal consistent sets with sets that also meet a certain further condition. Roughly speaking, that further condition is that whenever the set says that there *could* have been a possible individual, that possible individual also has a name. The difference between this method and what you might call the **method of actual names** is that we are using free logic instead of a classical logic. And so from the fact that a possible individual has a name, it need not follow that the possible individual in fact exists.

26. See Garson (2001) and chapter sixteen of Hughes and Cresswell (1996) for more on this.

Proving that $\mathbf{Q_2}$ is complete is structurally similar to proving that $\mathbf{K_2}$ is complete, with a few key modifications. For this reason, we are not going to go through the proof in anything like exhaustive detail. Instead, we are going to indicate points at which the completeness proof for $\mathbf{K_2}$ needs to be modified and show how those modifications can be made. This will all be familiar to those have used the method of possible names in the one-dimensional case.

Definition 10.1. *A set S is* ***strictly omega complete*** *when:*

> *If $\phi \dashv\!\!\exists\ (Ec \supset \psi c) \in S$ for every constant c that does not appear in $\phi \dashv\!\!\exists\ \forall x \psi x$, then $\phi \dashv\!\!\exists\ \forall x \psi x \in S$*

Definition 10.2. *A set of sentences is* ***superb*** *when it is maximal, consistent, and strictly omega complete.*

Once we have the notion of a superb set, the definition of a network is shifted accordingly. That is, instead of mapping pairs of worlds to maximal consistent sets, a network maps pairs of worlds to superb sets. The definitions of coherence and satisfaction and correspondence remain the same. What we need to do then is replace the old version of the truth lemma with a new one, since we now have quantified Kripke models instead of propositional Kripke models.

Lemma 10.1 (Truth Lemma)**.** *Every perfect network $\mathcal{N}$ has a corresponding Kripke model $\mathcal{M}$.*

Proof. Let $\mathcal{N} = \langle W, R, f \rangle$ be a perfect network assigning pairs of worlds to sets of sentences of $\mathcal{L}$. We then construct a corresponding model $\mathcal{M} = \langle W, R, D, d, [\![\cdot]\!] \rangle$ as follows. First, we define certain sets of terms:

$$\bar{c} = \{d \mid (c = d) \in f(wv) \text{ for any } w, v \in W\}$$

We then use these sets to construct $\mathcal{M}$:

$$\begin{aligned} D &= \{\bar{c} \mid c \in \mathcal{L}\} \\ d(wv) &= \{\bar{c} \mid Ec \in f(wv)\} \\ [\![c]\!] &= \bar{c} \\ [\![P]\!](wv) &= \{\langle \bar{c}_1, ..., \bar{c}_n \rangle \mid P(c_1, ..., c_n) \in f(wv)\} \end{aligned}$$

The result is that $\mathcal{M}$ is clearly a model. That it it corresponds to $\mathcal{N}$ can then be shown using an induction on the complexity of sentences. The main observation is that this requires both the preservation of identity and the preservation of distinctness, both of which are valid in $\mathbf{Q_2}$. ∎

Once we have the truth lemma, proving completeness is reduced to showing that every strongly consistent set has a perfect network. The first step in that process is proving following lemma, which is a kind of replacement for Lindenbaum's lemma.

Lemma 10.2. *Every consistent set of sentences S in language $\mathcal{L}$ can be extended to a superb set T in an extended language $\mathcal{L}^+$.*

Proof. Suppose that S is a consistent set of sentences in $\mathcal{L}$. We then extend the language $\mathcal{L}$ to the language $\mathcal{L}^+$ by adding countably many new constants. Since $\mathcal{L}^+$ has countably many sentences, we can fix an enumeration of those sentences and use it to build the following chain:

(1) $T_0 = S$

(2) If δ_{n+1} is of the form $\neg(\phi \strictif \forall x \psi x)$ and $T_n \cup \{\neg(\phi \strictif \forall x \psi x)\}$ is consistent, then $T_{n+1} = T_n \cup \{\neg(\phi \strictif \forall x \psi x), \neg(\phi \strictif (Ec \supset \psi c))\}$, where c is a constant that has not yet appeared in the chain.

(3) If δ_{n+1} is not of the form $\neg(\phi \strictif \forall x \psi x)$ and $T_n \cup \{\phi\}$ is consistent, then $T_{n+1} = T_n \cup \{\phi\}$.

(4) If δ_{n+1} is not of the form $\neg(\phi \strictif \forall x \psi x)$ and $T_n \cup \{\phi\}$ is inconsistent, then $T_{n+1} = T_n \cup \{\neg\phi\}$.

(5) $T = \bigcup T_i$

We then claim that T is a superb set extending S. To show this, we need to show that S is a subset of T, that T is maximal, that T is consistent, and that T is strictly omega complete. The interesting case is showing that the first recursive clause preserves consistency. To that end, suppose that $T_n \cup \{\neg(\phi \strictif \forall x \psi x)\}$ is consistent and let c be any constant not appearing in that set. Now suppose for reductio that

$$T_n \cup \{\neg(\phi \strictif \forall x \psi x)\} \cup \{\neg(\phi \strictif (Ec \supset \psi c))\}$$

is inconsistent. In that case, there are $\gamma_1, ..., \gamma_m \in T_n$ such that

$$(\gamma_1 \wedge ... \wedge \gamma_m \wedge \neg(\phi \strictif \forall x \psi x)) \supset (\phi \strictif (Ec \supset \psi c))$$

But this sentence itself is a strict condition and c appears nowhere outside $Ec \supset \psi c$. So by MUG and predicate logic and the fact that all of our intensional operators are normal:

$$(\gamma_1 \wedge ... \wedge \gamma_m \wedge \neg(\phi \strictif \forall x \psi x)) \supset (\phi \strictif \forall x \psi x)$$

But then by propositional logic:

$$\neg(\gamma_1 \wedge ... \wedge \gamma_m)$$

So the original set is inconsistent, which is contrary to assumption. ■

Once we have lemma 10.2, we can show that every strongly consistent set has a coherent network using the same argument as before. We then need to prove a new version of the repair lemma. The new proof turns out to be the same as the old proof, with the exception that we need to appeal to lemma 10.3 in place of lemma 9.3. This is need to ensure that we can always add a world as an appropriate witness without further expanding the language.

Lemma 10.3. *If there is a superb set of sentences S and $\Diamond\phi \in S$, then there is a superb set of sentences extending $T_0 = \{\psi \mid \Box\psi \in S\} \cup \{\phi\}$ in the same language as S.*

Proof. Let S be a superb set of sentences. We then construct a superb set of sentences T extending T_0 by building a chain of sets just like the one in lemma 10.2. The only difference is that we set $T_0 = \{\psi \mid \Box\psi \in S\} \cup \{\phi\}$ and do not expand the language. The main concern, then, is that the first recursive clause is not well defined, since there may not be any such constant c. What we need to show is that this concern is misplaced.

Suppose for reductio that there is no such c. That is, suppose that $T_n \cup \{\neg(\phi \dashv\exists \forall x\psi x)\}$ is consistent, but that no such T_{n+1} is consistent. This means that for every c, there are $\Box\gamma_1, ..., \Box\gamma_n \in S$ such that:

$$(\gamma_1 \wedge ... \wedge \gamma_n) \wedge \neg(\phi \dashv\exists \forall x\psi x) \supset (\phi \dashv\exists (Ec \supset \psi c))$$

It then follows by the normality of the box operator that

$$(\Box\gamma_1 \wedge ... \wedge \Box\gamma_n) \supset \Box(\neg(\phi \dashv\exists \forall x\psi x) \supset (\phi \dashv\exists (Ec \supset \psi c)))$$

for all constants c. But then since

$$\Box\gamma_1 \wedge ... \wedge \Box\gamma_n \in S$$

and S is closed under logical entailment, we have

$$\Box(\neg(\phi \dashv\exists \forall x\psi x) \supset (\phi \dashv\exists (Ec \supset \psi c))) \in S$$

for all c. Because S is strictly omega complete, this entails that:

$$\Box(\neg(\phi \dashv\exists \forall x\psi x) \supset (\phi \dashv\exists \forall x\psi x)) \in S$$

But then

$$\neg(\phi \dashv\exists \forall x\psi x) \supset (\phi \dashv\exists \forall x\psi x) \in T_n$$

which contradicts the original assumption that $T_n \cup \{\neg(\phi \dashv\exists \forall x\psi x)\}$ is consistent. So there is always a constant c of the kind required. ∎

Once we have a new version of the repair lemma, it follows that every strongly consistent set has a perfect network by lemma 9.5, which we proved in the last section. But then we have the result that we wanted.

Theorem 10.1 (Completeness)**.** *Every strongly* $\mathbf{Q_2}$*-consistent set has a two-dimensional quantified Kripke model.*

Proof. Immediate from the preceding. ∎

11. Simplifying Matters

In the last section, we shoed that $\mathbf{Q_2}$ is complete using MUG and PD. What we are going to do now is show that these principles can be derived in $\mathbf{QTB_2}$, and so do not need to be taken as basic. This is the major step toward showing that $\mathbf{QTB_2}$ is complete with respect to the class of reflexive and symmetric frames. The rest of the argument is just a reprisal of the last two section.

Lemma 11.1. *The following rules are all valid in* $\mathbf{QTB_2}$.

Export $\phi \supset \Box\psi \Rightarrow \otimes\Diamond\otimes\phi \supset \psi$

$\phi \supset \otimes\psi \Rightarrow \otimes\phi \supset \psi$

$\phi \supset \uparrow\psi \Rightarrow \phi \supset \Box(\mathsf{Id} \supset \psi)$

Import $\otimes\Diamond\otimes\phi \supset \psi \Rightarrow \phi \supset \Box\psi$

$\otimes\phi \supset \psi \Rightarrow \phi \supset \otimes\psi$

$\phi \supset \Box(\mathsf{Id} \supset \psi) \Rightarrow \phi \supset \uparrow\psi$

Proof. Using TB, we can show the following, then use it to derive both the first export rule and the first import rule:

$$\otimes\Diamond\otimes\Box\phi \supset \phi$$

The second export rule and the second import rule are immediate by A2 and A4. The third import rule follows from T, which can be derived from TB. And finally, the third export rule can be derived using $\mathbf{K_2}$. The basic idea is to first show that

$$\Box\otimes\uparrow\phi \supset \Box\otimes(\mathsf{Id} \supset \phi)$$

using A7 and the fact that the relevant operators are normal. A5 and propositional logic then give us:

$$\phi \supset \Box\otimes(\mathsf{Id} \supset \phi)$$

At this point, we just have to eliminate the swap operator from the consequent. This can be done by showing that

$$\otimes(\mathsf{Id} \supset \phi) \equiv (\mathsf{Id} \supset \phi)$$

and using substitution. We then have

$$\uparrow\phi \supset \Box(\mathsf{Id} \supset \psi)$$

But in that case, we also have the validity of the third export rule. ■

Lemma 11.2. *MUG is valid in* $\mathbf{QTB_2}$.

Proof. Take any strict conditional. We can assume that it has no leading operators because, if it did, we could just use the logically equivalent $\top \supset (\phi \prec\!\!\!3\ \psi)$. This means that we now have a sentence

$$\phi_1 \supset \ldots (\phi_2 \supset \ldots \psi \ldots)$$

with any number of operators in front of the first embedded conditional. For concreteness, we can think of this as:

$$\phi_1 \supset \Box\otimes\downarrow(\phi_2 \supset \ldots \psi \ldots)$$

Repeated applications of the export rules let us move the operators out of the consequent.

$$\Diamond\otimes\otimes\Diamond\otimes\phi_1 \supset (\mathsf{Id} \supset (\phi_2 \supset \ldots \psi \ldots))$$

Propositional logic then gives us:

$$(\Diamond\otimes\otimes\Diamond\otimes\phi_1 \wedge \mathsf{Id} \wedge \phi_2) \supset \ \dots (\dots \psi \dots)$$

Now at this point, there could be any of number of further embedded conditions with any number of operators in front of them. But if so, we can just repeat the process just described until there are none. We can assume, then, without loss of generality, that there are no further embedded conditionals. This gives us:

$$(\Diamond\otimes\otimes\Diamond\otimes\phi_1 \wedge \mathsf{Id} \wedge \phi_2) \supset \psi$$

But now what we have is just an ordinary material conditional. So we can universally generalize using predicate logic.

$$(\Diamond\otimes\otimes\Diamond\otimes\phi_1 \wedge \mathsf{Id} \wedge \phi_2) \supset \forall x\psi$$

Propositional logic and repeated application of the import rules then let us put all the operators back where we found them.

$$\phi_1 \supset \Box\otimes\downarrow(\phi_2 \supset \forall x\psi)$$

So MUG is valid in $\mathbf{QTB_2}$. ■

Lemma 11.3. *PI and $PD_\otimes$ are valid in* $\mathbf{FK_2}$*. Moreover, $PD_\uparrow$ entails $PD_\Box$ in* $\mathbf{FK_2}$ *and $PD_\uparrow$ is valid in* $\mathbf{QTB_2}$.

Proof. The proof that $\mathrm{PI}_\Box$ is valid is the same as in the one-dimensional case. Since that proof only requires QPL and RN, the other two cases are the same. We can then derive $\mathrm{PD}_\otimes$ from $\mathrm{PI}_\otimes$ and A2 using propositional logic. This leaves us with only two remaining proofs.

To show: $t \neq s \supset \uparrow(t \neq s) \Rightarrow t \neq s \supset \Box(t \neq s)$

(1) $(t \neq s) \supset \uparrow(t \neq s)$

(2) $\uparrow(t \neq s) \supset \Box\otimes\uparrow(t \neq s)$ A5

(3) $(t \neq s) \equiv \uparrow(t \neq s)$ 1, PI, A1

(4) $(t \neq s) \equiv \otimes(t \neq s)$ $PD_{\otimes}$, PI, A2

(5) $(t \neq s) \supset \Box(t \neq s)$ 2, 3, 4, sub

To show: $t \neq s \supset \uparrow(t \neq s)$

(1) $\uparrow(t = s) \supset \uparrow(t = s)$ PL

(2) $\uparrow(t = s) \supset \uparrow\Box\otimes(t = s)$ 1, PI

(3) $\uparrow(t = s) \supset \Box\otimes(t = s)$ 2, A6

(4) $\uparrow(t = s) \supset (t = s)$ 3, TB

(5) $(t \neq s) \supset \uparrow(t \neq s)$ 4, PL, A1

■

Theorem 11.1 (Completeness)**.** *Every* **$\mathbf{QTB_2}$**-*consistent set has a two-dimensional Kripke model.*

Proof. By lemma 11.2 and lemma 11.3 and an argument like the one from section 7 ■

Earlier, we noted that you might want to drop the Id operator. If so, then the derivation of MUG in lemma 11.2 will not go through, since the export rule for ↑ makes essential use of Id. Fortunately, when identity is dropped, there is another strategy available. This second strategy relies on accepting $BF_{\uparrow}$. This axioms says, basically, that everything that exists at a world w from its own perspective also exists at w from the perspective of any other world v.

Lemma 11.4. *The following rules are all valid in* $\mathbf{QBT_2}$ *when the identity operator is dropped from the language.*

Export $\phi \supset \Box\psi \Rightarrow \otimes\Diamond\otimes\phi \supset \psi$

$\phi \supset \otimes\psi \Rightarrow \otimes\phi \supset \psi$

$\phi_1 \supset \uparrow(\phi_2 \supset \psi) \Rightarrow \phi_1 \supset (\uparrow\phi_2 \supset \uparrow\psi)$

$\phi \supset \uparrow\Box\phi \Rightarrow \phi \supset \Box\phi$

$\phi \supset \uparrow\otimes\phi \Rightarrow \phi \supset \uparrow\phi$

$\phi \supset \uparrow\uparrow\phi \Rightarrow \phi \supset \uparrow\phi$

Import $\otimes\Diamond\otimes\phi \supset \psi \Rightarrow \phi \supset \Box\psi$

$\otimes\phi \supset \psi \Rightarrow \phi \supset \otimes\psi$

$\phi_1 \supset (\uparrow\phi_2 \supset \uparrow\psi) \Rightarrow \phi_1 \supset \uparrow(\phi_2 \supset \psi)$

$\phi \supset \Box\phi \Rightarrow \phi \supset \uparrow\Box\phi$

$\phi \supset \uparrow\phi \Rightarrow \phi \supset \uparrow\otimes\phi$

$\phi \supset \uparrow\phi \Rightarrow \phi \supset \uparrow\uparrow\phi$

Proof. Immediate by lemma 11.1 and the basic rules and axioms. ■

Lemma 11.5. *MUG is valid in* $\mathbf{QTB_2}$ *when the identity operator is dropped and the axiom* $BF_{\uparrow}$ *is added.*

Proof. As before, take any strict conditional. We can assume that it has no leading operators, for the reason described earlier. We thus have a sentence of the form

$$\phi_1 \supset \ldots (\phi_2 \supset \ldots \psi \ldots)$$

with any number of operators in front of the first embedded conditional. For concreteness, we can think of this as:

$$\phi_1 \supset \uparrow\otimes\uparrow\Box\uparrow(\phi_2 \supset \ldots \psi \ldots)$$

Our new export rules let us eliminate all the arrows and swaps in front of the box.

$$\phi_1 \supset \Box\uparrow(\phi_2 \supset \ldots \psi \ldots)$$

The export rule for the box lets us move it out of the way as well.

$$\otimes\Diamond\otimes\phi_1 \supset \uparrow(\phi_2 \supset \ldots \psi \ldots)$$

Now we have an up operator directly in front of a material condition. Using the third export rule, we can distribute this operator. Propositional logic then gives us:

$$(\otimes\Diamond\otimes\phi_1 \wedge \uparrow\phi_2) \supset \uparrow \ldots (\ldots \psi \ldots)$$

At this point, the inner $\uparrow$ operator may be prefacing any number of operators. There may also be any number of further embedded conditionals. But in either case, we can just repeat the process just described to remove them. So we can assume, without loss of generality, that there are no further operators or embedded material conditionals. The result is:

$$(\otimes\Diamond\otimes\phi_1 \wedge \uparrow\phi_2) \supset \uparrow\psi$$

At this point, we can use predicate logic to universally generalize and move the universal quantified directly outside the innermost up operator.

$$(\otimes\Diamond\otimes\phi_1 \wedge \uparrow\phi_2) \supset \forall x\uparrow\psi$$

Using the axiom $\mathrm{BF}_{\uparrow}$, we can then move that quantifier inside the up operator.

$$(\otimes\Diamond\otimes\phi_1 \wedge \uparrow\phi_2) \supset \uparrow\forall x\psi$$

At this point, we can then use the import rules to put all the operators back where we found them. The result is:

$$\phi_1 \supset \uparrow\otimes\uparrow\Box\uparrow(\phi_2 \supset \forall x\psi)$$

So MUG is valid in $\mathbf{QTB_2}$ when the identity operator is dropped from the language and the axiom $\mathrm{BF}_{\uparrow}$ is added. ■

Theorem 11.2 (Completeness). *Every* $\mathbf{QTB_2}$*-consistent set has a two-dimensional Kripke model when the axiom* $BF_{\uparrow}$ *is added and the* Id *operator is dropped.*

Proof. By lemma 11.3 and lemma 11.5 and an argument like the one from section 7 ■

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